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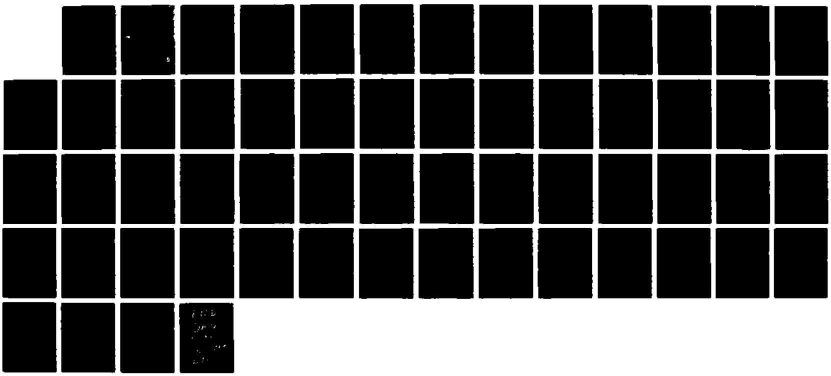

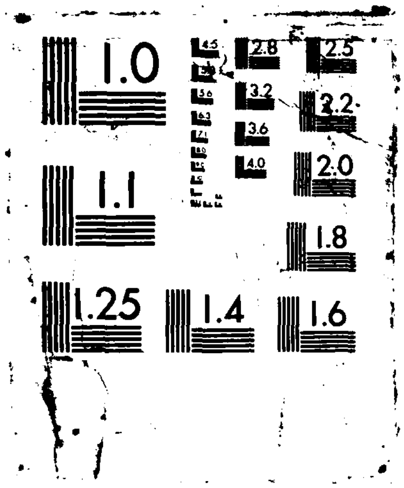
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TIME-OPTIMAL ADAPTIVE SHAPED TORQUE (TOAST) TECHNICAL NOTE

D. L. Cate

January 1988

Final Report

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AIR FORCE WEAPONS LABORATORY
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A discussion of time-optimal retargeting of beam steering mirrors is given. The argument for a bang-bang waveform is given first along with a simple 2nd-order regulator problem as an example. The method of solution for a 2nd-order tracker problem is given with an example. An argument is then put forth for using the time-optimal bang-bang method on closed-loop control systems. The solution for the general multiple-order complex eigenvalue bang-bang boundary value problem is derived and an example is shown. This is followed by the demonstration of the time savings of the time-optimal retargeting technique, in which a lead compensator with a pure inertia is used to show the relationship between the ratio of maximum torque to the torque required by a classical set-point change and the savings in time for general retargeting problems. Finally, the time-optimal retargeting solution is applied to a 9th-order model of an existing beam steering mirror. From this analysis several conclusions about using time-optimal control on real systems are drawn.

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FOREWORD

Time-optimal beam steering requires a type of control that is much different than conventional methods. The typical controller design is one in which states of the system are fed back against some incoming control signal designed to control a certain state. The design of the controller centers on the method of feedback and usually some pole placement technique which makes the system assume the desired characteristics. But time-optimal control is concerned with neither pole placement nor method of feedback. Time-optimal control is concerned solely with designing the input control signal to make the system attain a desired end condition in the fastest possible time. The only design criterion is time, and the only part of the system affected is the input.

The form of time-optimal control postulated in this series is bang-bang. The bang-bang waveform switches between the maximum and minimum control values with precisely the right frequency and duration to bring the system's model from its initial condition to a desired end condition.

Time-optimal control is an open-loop method of control, because the control waveform is calculated from a priori boundary conditions and does not require adaptation during the exercise of the control. Nevertheless, it can be used on a closed-loop system because it is performed and calculated independently of the system's controller. The time-optimal control is calculated by modeling the closed-loop system in the time-domain and then solving a boundary-value problem for the entire system, including the control.

Time-optimal control offers an advantage for speed of retargeting and stabilization over conventional methods of input. In fact, the more rigorous the settling criteria, the better time-optimal control performs, because it is solved for boundary conditions (initial condition and final desired condition). It provides the fastest movement from one state to another to whatever degree of accuracy the model's parameters allow. On the other hand, classical control simply uses a constant input for retargeting and lets the system's dynamics cancel the error at the characteristic rate of the system. Because of this, accurate settling criteria take a great deal of time to

achieve classically, whereas the time needed to retarget optimally is relatively insensitive to accuracy requirements.

The time-optimal problem can be solved for any system that can be modeled in the time domain. The solution for a beam steering problem typically can be guessed at, in general, to require n-1 switchings between maximum and minimum command values, where n is the system order. This seems to hold true for small displacement steerings from zero initial conditions, a problem consistent with optical retargeting. This boundary-value problem can be solved for any size system by reducing the order of the model and gradually building up to the full-order model and exact solution by using solutions obtained from the simpler lower-order models.

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I. OVERVIEW

This report in the Time-Optimal Adaptive Shaped Torque (TOAST) effort was originally five separate papers, written over 2 years, describing the progressive breakthroughs in the understanding of the time-optimal bang-bang problem. At first, very little was known about any method of solving time-optimal problems. At first, simple second-order problems were solved. The major turning point for understanding of the bang-bang problem was the discovery of a general method of solution that works for large-order models with complex eugenvalues. From there, some general conclusions were made about time-optimal enhancement. Finally, time-optimal problems were solved for a 9th order beam steering mirror (BSM) model.

Section II was the first TOAST article written. It outlines the common variational calculus for the time-optimality of the bang-bang waveform. It also describes the graphical switching curve method for solving simple second-order tracker (nonzero end condition) problems. The methods of solution here are good only for second order problems. Nevertheless, they are a step in understanding bang-bang theory.

Section III explains how the time-optimal control can be used with a closed-loop system and why it is desirable to do so. It begins with a derivation to show why maximum commands yield such an improvement in retargeting speed. It then gives some simple second-order examples, comparing time-optimal effects on closed- and open-loop systems.

Section IV is an adaptation from the most critical article of the entire TOAST series. This paper (presented at the 1986 American Control Conference in Seattle in June 1986) derives the general solution of the bang-bang boundary value problem for n^{th} order systems with complex eigenvalues. On this algorithm, the time-optimal solver code, SYS, is based. This section explains the algorithm and gives a simple third-order example.

Section V quantitatively describes the enhancement of a Type II controller using time-optimal techniques. It shows how the effective retargeting

bandwidth of a system is improved with time-optimal control and that this enhancement becomes more dramatic as the settling requirements for the system become more rigorous. This section also details the canonical time-domain form for a system model and the time and magnitude scaling that are essential to successfully using the SYS algorithm.

Section VI describes the solution of time-optimal problems for the 9^{th}-order model of an actual BSM in the laboratory. It begins by describing the 9^{th}-order transfer function for position versus control input. It then explains the time-domain modeling of the system along with the necessary scaling approaches. Then, it details a method of solving large-order problems through reduction of order and gradual rebuilding to an exact solution for a full-state model. Most importantly, it introduces the apparent one-for-one relationship between system eigenvalues and the associated time legs of the exact time-optimal control solution. The report concludes with a discussion of the performance of lower-order models, arguing that the advantages to time-optimal control may be obtained by simply solving for reduced-order models that account for the system's *critical* eigenvalues instead of the more complicated full-order models.

II. TRACKER FORMULATION OF THE BANG-BANG PROBLEM

TIME-OPTIMALITY OF THE BANG-BANG TECHNIQUE

Using Hamiltonian analysis and the maximum principle, we will demonstrate the time-optimality of the bang-bang approach. We begin with a system for which the dynamics and control are described as

$$\dot{\bar{x}} = A\bar{x} + \bar{b}u \tag{1}$$

where A is the dynamics matrix, $\bar{b}$ is the input vector and u is the scaler control. The control variable u, for most physical systems, is constrained to some maximum amplitude. For instance, if u describes an input torque, there is a maximum limit to the magnitude of the torque in either positive or negative sense. So we have limits on u, a and a , such that $a_1 \leq u \leq a_2$.

To invoke variational methods, the time-optimal cost function is defined, which in its Lagrange form for an unspecified time problem is

$$J = \int_{t_0}^{t_f} \phi(\bar{x},\dot{\bar{x}},t)\, dt \tag{2}$$

For the time-optimal problem, $\phi = 1$ or $> J = T_f - t_0$. Using the method of Lagrange multipliers, we have

$$J = \int_{t_0}^{t_f} \left[1 + \bar{\lambda}^T(Ax + \bar{b}u - \dot{\bar{x}})\right] dt \tag{3}$$

Defining the Hamiltonian $H = 1 + \bar{\lambda}^T(A\bar{x} + \bar{b}u)$ (Ref. 2), we then have

$$J = \int_{t_0}^{t_f} \left[H - \bar{\lambda}^T\dot{x}\right] dt \tag{4}$$

Taking the variation of J and setting it to zero, we have

$$\delta J = \int_{t_o}^{t_f}\left[\left(\frac{\partial H}{\partial \bar{x}} - \bar{\lambda}^T\right)\delta\bar{x} + \left(\frac{\partial H}{\partial \lambda} - \dot{\bar{x}}\right)\delta\bar{\lambda} + \frac{\partial H}{\partial u}\delta u\right]dt = 0 \tag{5}$$

After integration by parts,

$$J = -\bar{x}\bar{\lambda}^T\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f}\left[\left(\frac{\partial H}{\partial \bar{x}} + \dot{\bar{\lambda}}^T\right)\delta\bar{x} + \left(\frac{\partial H}{\partial \lambda} - \dot{\bar{x}}\right)\delta\bar{\lambda} + \frac{\partial H}{\partial u}\delta u\right]dt = 0 \tag{6}$$

From this the canonical equations are

$$\frac{\partial H}{\partial u} = 0 \qquad \frac{\partial H}{\partial \bar{\lambda}} = \dot{\bar{x}} \qquad \frac{\partial H}{-\partial \bar{x}} = \dot{\bar{\lambda}}^T \tag{7}$$

Because nothing is known about $\bar{\lambda}$, these equations are not helpful in determining an optimum u except for the fact that the Hamiltonian must be minimized with respect to u. Because $H = 1 + \bar{\lambda}^T(A\bar{x} + \bar{b}u)$, it is clear that H is minimized for $a_1 \leq u \leq a_2$ when

$$\begin{aligned} u &= a_1 \text{ for } \bar{\lambda}^T\bar{b} > 0 \\ u &= a_2 \text{ for } \bar{\lambda}^T\bar{b} < 0 \end{aligned} \tag{8}$$

and thus this defines the optimal control.

The variable $\bar{\lambda}$, called the influence function, is useful for some bang-bang problems. From Equation 7, we know that $\dot{\bar{\lambda}} = -A^T\bar{\lambda}$, hence, the influence function is characterized by the same dynamics as the uncontrolled $\bar{x}$. An expression for $\bar{\lambda}$ can be determined in terms of $\bar{\lambda}$ at the final state. As will be seen, u is an explicit function of $\bar{\lambda}$ and so the control law may be expressed in terms of a final condition for $\bar{\lambda}$. This relationship is useful for finding the set of possible initial conditions given a time interval for the problem (Ref. 1). However, for these purposes we have known initial conditions,

therefore, $\bar{\lambda}$ is not a useful variable to determine. In the second-order case, it is necessary only to know the system dynamics and the boundary conditions to do the problem.

THE BANG-BANG REGULATOR

We can now specify the time-optimal or bang-bang control for a regulator constraining u to be $|u| \leq 1$ and thus, $u = -\text{sign}\left[\bar{\lambda}^T\bar{B}\right]$. For the two-dimensional problem, (i.e., one in which only two states are defined) the bang-bang problem can be solved graphically by the use of a switching curve. To determine the switching curve, it is necessary to integrate backwards in time from the origin. If we fine the switching curve state as $\bar{\xi}$, we have

$$\dot{\bar{\xi}} = -A\bar{\xi} - \bar{B}u_f \tag{9}$$

where u_f is the final control value which will be used to drive the state to zero. Shown in Fig. 1 is a switching curve, in phase-space (x1 - x2), for the problem:

$$\dot{\bar{x}} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{10}$$

In the second-order case, the final control value is the second control value used. Figure 2 shows the time-optimal trajectory given an initial condition of $\begin{bmatrix} 2 & -1 \end{bmatrix}^T$.

Once the switching curve is drawn in two dimensions, one only has to propagate the system forward in time, from any initial condition, with a value for u which will impel the state toward the switching curve. When the state intercepts the switching curve, the value of the control is switched to the value of the opposite sign, causing the system to follow the switching curve to the origin. If this procedure is followed as demonstrated in Fig. 2, the result is the optimum switching law for any second order system.

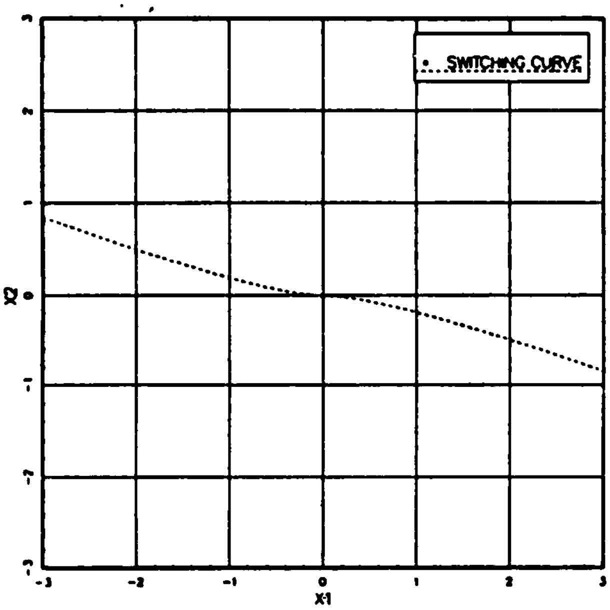


Figure 1. Regulator Switching Curve.

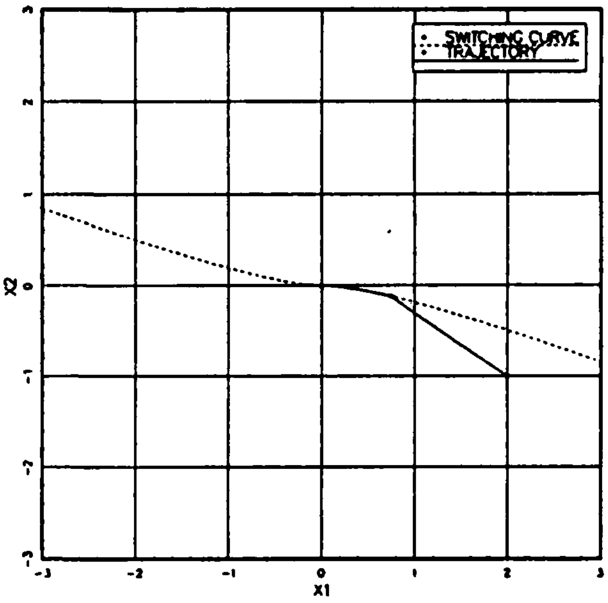


Figure 2. Trajectory of Regulator Example.

The time elapsed to the switching from u = 1 to u = -1 (the switching time) is 0.693 s, and the time from the switch to the origin is 0.405 s for an approximate total time of 1.098 s. It is simple to calculate that this same displacement for the command input u = 0 takes 3.4 s in total time to reach the origin (within a tolerance of 0.1). We see then that, to the extent that the maximum magnitude of the input commands are greater than the input commands required for the desired displacement, the bang-bang technique has a dramatic effect in terms of time-optimality.

Larger order systems can be regulated using more switchings. It has been shown that an n^{th}-order system with real poles can be brought to a zero state in the shortest time using the bang-bang method with a maximum of n-1 switchings (Ref. 2). The switching curve, however, is only good for indicating the final switch. Calculation of the input bang-bang profile for systems of order larger than 2 must be done by diagonalizing the system with a linear transformation and representing the intersection of the switching hypersurfaces with a series of n nonlinear equations (Ref. 3). The solution of this problem will be detailed in Section IV.

THE BANG-BANG TRACKER

For purposes related to space systems that require a fast, accurate displacement with no ringing, the tracker formulation of the bang-bang problem is necessary. Rather than reducing the state $\bar{x}$ to zero, we wish to bring the state $\bar{x}$ to some target set of conditions $\bar{x}_0$. We can define the error of state $\bar{x}$ to be $\bar{y} = \bar{x} - \bar{x}_0$. We then have:

$$\begin{aligned} \dot{\bar{y}} &= \dot{\bar{x}} \\ \dot{\bar{x}} &= A\bar{x} + \bar{b}u \\ \bar{x} &= \bar{x}_0 + \bar{y} \\ \dot{\bar{y}} &= A\bar{y} + A\bar{x}_0 + \bar{b}u \end{aligned} \tag{11}$$

The solution to this problem follows the same course as the solution for the tracker except that now the equation for the switching curve is

$$\dot{\bar{\xi}} = -A\bar{\xi} - A\bar{x}_0 - \bar{b}u_f \tag{12}$$

The switching originates from the origin because $\bar{\xi}$ is a variable of y-space, and we wish to bring the error to zero. To construct the switching curve in $\bar{x}$-space, the transformation $\bar{x} = \bar{\zeta} + \bar{x}_0$ is used. As a result of this transformation, the x-space switching curve originates from $\bar{x}_0$ and is symmetric about this point as shown below in Fig. 3 for which the target set is $\bar{x}_0 = \begin{bmatrix} 0 & 0.25 \end{bmatrix}^T$.

The problem is solved in the same manner as the regulator. An initial condition is chosen, in this case, $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$. The system is then propagated from this initial condition until the switching curve is struck. A switch in the control is then mandated and the states are brought to the desired condition.

For this problem, the switching time is 1.171 s, and 0.119 s elapse from the switching point to the origin for a total of 1.290 s. Using the input command u - 0.5 to bring the system to the desired state requires over 3.5 s (for a 5 percent settling time). Again, the bang-bang method renders a considerable time savings.

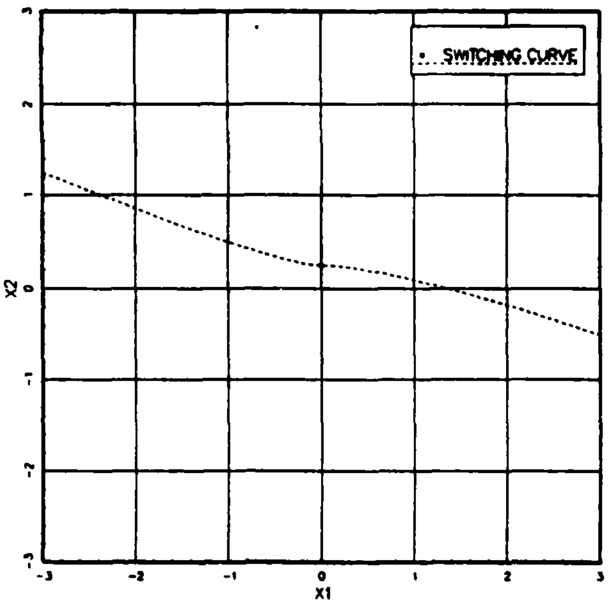


Figure 3. Tracker Switching Curve.

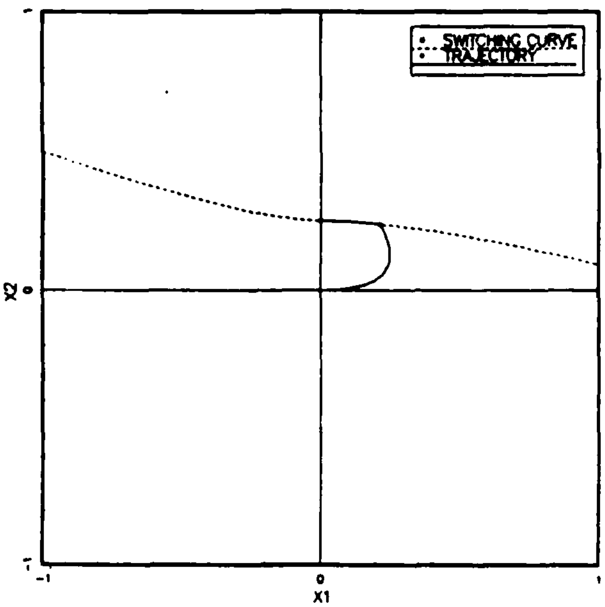


Figure 4. Trajectory of Tracker Example.

CONCLUSION

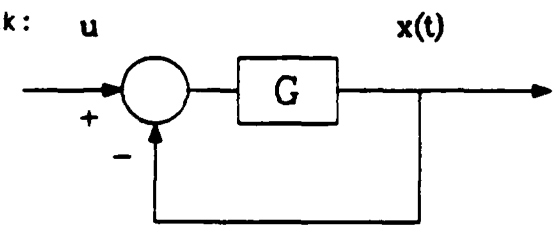
The bang-bang problem for a second-order system (with real poles) is a relatively simple one which can be solved graphically using the technique of the switching curve. The bang-bang approach renders the time-optima) solution to any problem for which the maximum control value is greater than that required to command the system to a target state by conventional means. This method provides a substantive savings in time.

III. BANG-BANG CONTROL OF CLOSED-LOOP VERSUS OPEN-LOOP SYSTEMS

MOTIVATION

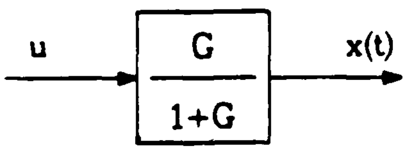
The option of controlling a closed or open loop is a critical issue in the use of bang-bang control for BSMs. The time response of a closed-loop system is highly dependent on the value of the input. The attractiveness of the bang-bang approach to control hinges on the fact that the fastest way to make a system respond is to give it maximum commands as opposed to giving it the desired value for the state as the input and waiting for the system dynamics to eliminate the error. The bang-bang technique will always work faster as long as the maximum inputs are of larger magnitude than those used to command the system to attain a state in the classical way. This is true for closed-loop and open-loop systems.

We can examine a simple second-order system and close the loop with unity feedback:



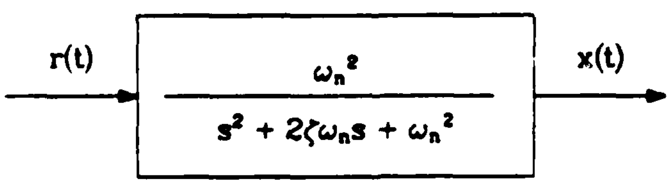
$$G = \frac{1}{(s+p_1)(s+p_2)}$$

This system simplifies easily to



$$\frac{G}{1+G} = \frac{1}{s^2 + (p_1 + p_2)s + (p_1p_2 + 1)}$$

If the system has a complex conjugate pair for its poles, then we have the familiar representation:



where ζ is the damping and ω_n is the natural frequency. In state space:

$$\dot{x} = \begin{bmatrix} -2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad (13)$$

where $u = \omega_n^2 r$, $x_2 = x$, and $x_1 = \dot{x}_2 = \dot{x}$.

We can go back to the differential equation in $x(t)$ for this closed-loop system and solve it in time. By inspection of the block diagram:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = r\omega_n^2 \tag{14}$$

Now if r is a constant (which it is for this purpose), then:

$$x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + r \tag{15}$$

where

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \text{ or } -\zeta\omega_n \mp i\omega_n\sqrt{1 - \zeta^2} \tag{16}$$

The eigenvalues (λ_i) are dependent solely on the characteristics of the system. However, solving for c_1, and c_2 given initial condition x_o and $\dot{x}_o$, we have:

$$c_1 + c_2 = x_o - r \qquad \lambda_1c_1 + \lambda_2c_2 = \dot{x}_o$$

$$c_1 = \frac{(x_o - r)\lambda_2 - \dot{x}_o}{\lambda_2 - \lambda_1} \qquad c_2 = \frac{\dot{x}_o + \lambda_1(r - x_o)}{\lambda_1 - \lambda_2} \tag{17}$$

Hence,

$$x(t) = \left[\frac{\dot{x}_o - x_o\lambda_2}{2\omega_n\sqrt{\zeta^2 - 1}}\, e^{\lambda_1 t} + \frac{x_o\lambda_1 - \dot{x}_o}{2\omega_n\sqrt{\zeta^2 - 1}}\, e^{\lambda_2 t}\right] + \left[\frac{\lambda_2e^{\lambda_1 t} - \lambda_1e^{\lambda_2 t}}{2\omega_n\sqrt{\zeta^2 - 1}} + 1\right] r \tag{18}$$

Now we have found conclusively that the time response for a general second-order closed-loop system is directly related to the value of the input value r--that not only is the steady-state value of the response changed, but also the transient response is changed.

Furthermore, if the initial conditions are $x_o = \dot{x}_o = 0$, as they typically are for this purpose,

$$x(t) = \left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta \sin\,(\omega_n\sqrt{1-\zeta^2})t + \sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2})t\right]\right)r \qquad (19)$$

which is, of course, directly proportional to r. If we make the assumption that the state will settle at a time close to zero (good for this purpose), we have:

$$x\,(t) \sim [1-e^{-\zeta\omega_n t}]\; r \qquad (20)$$

If rise time, t_r, is defined as the time it takes for the system to go from 0.1 to 0.9 of the desired steady-state value (Ref. 4), we see that the rise time is dependent both on the system's poles and the input r. If a higher input r is used, the closed-loop system will move toward the target more quickly. From this intuitive argument, it can be seen that the time-optimal approach to attaining the desired steady-state value is to use maximum inputs (bang-bang). For example, assume that the maximum possible input for r is 2.0, and desired steady-state value for x is 1.0. Using the conventional method, we would input r = 1 so that

$$x(t) = 1-e^{-\zeta\omega_n t} \qquad (21)$$

Because rise time, t_r, is defined as the time it takes to get from x = 0.1 to x = 0.9, define t_1 as the time it takes to get from x = 0 to x = 0.1 and t_2 as the time it takes to attain x = 0.9. We have

$$1-e^{-\zeta\omega_n t_1} = 0.1 \Rightarrow -\zeta\omega_n t_1 = \ln(0.9) \Rightarrow t_1 = \frac{0.105}{\zeta\omega_n}$$

$$1-e^{-\zeta\omega_n t} = 0.9 \Rightarrow -\zeta\omega_n t_2 = \ln(0.1) \Rightarrow t_2 = \frac{2.303}{\zeta\omega_n}$$

$$t_r = t_2 - t_1 \sim \frac{2.2}{\zeta\omega_n} \qquad (22)$$

If, however, r = 2 then

$$x(t) + 2\text{-}2e^{-\zeta\omega_n t}$$

$$1-e^{-\xi\omega_n t_1} = 0.05 \Rightarrow t_1 = \frac{0.0513}{\zeta\omega_n} \qquad 1-e^{-\xi\omega_n t_2} = 0.45 \Rightarrow t_2 = \frac{0.598}{\zeta\omega_n} \quad (23)$$

and

$$t_r \sim \frac{0.55}{\zeta\omega_n}$$

which is one-fourth of the rise time for the input of r = 1. Hence, the use of maximum inputs has a great effect on the speed of the closed-loop system's reaction and it will be seen in the examples that the bang-bang technique will have a dramatic effect on the response for both closed-loop and open-loop systems.

EXAMPLE 1: CLOSED-LOOP SYSTEM

Because the purpose of this section is to demonstrate bang-bang performances for a system given different conditions, only results will be given. To define a regulator problem, consider a real-pole system with a closed-loop characteristic equation $s^2 + 3s + 2$. If this is a closed-loop system with unity feedback, the closed-loop poles are $\lambda_1 = -1$, and $\lambda_2 = -2$, and the open-loop poles are $p_1 = -0.3820$ and $p_2 = -2.618$. Thus, the closed-loop system is

$$\dot{\bar{x}} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad (24)$$

where $|u| \leq 1$ and the target conditions is $\bar{x} = [0\ 0]^T$. If we define the initial condition as $\bar{x}_0 = [2\ -1]^T$, then the time-optimal control is

$$u = 1 \qquad 0 \leq t \leq 0.693$$

$$u = -1 \qquad 0.693 \leq t \leq 1.098 \qquad (25)$$

Hence, in approximately 1.098 s the system is brought to zero as shown in Fig. 5.

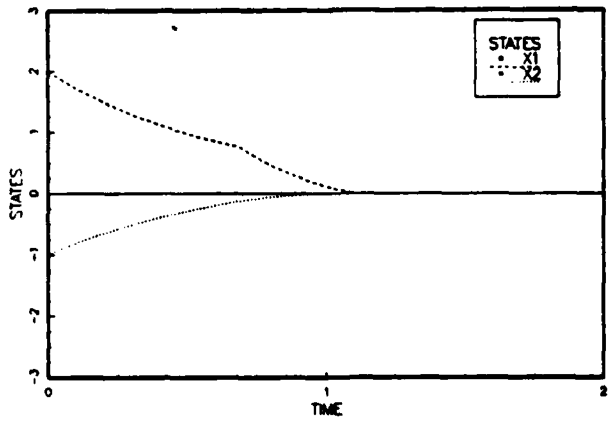


Figure 5. Time-Optimal Time Response of Example 1.

The break point in the trajectory shows where the control switch was made. If we were to use the conventional command input technique (which uses the system's natural dynamics to reduce the error) then we would input u = 0 as shown in Fig. 6.

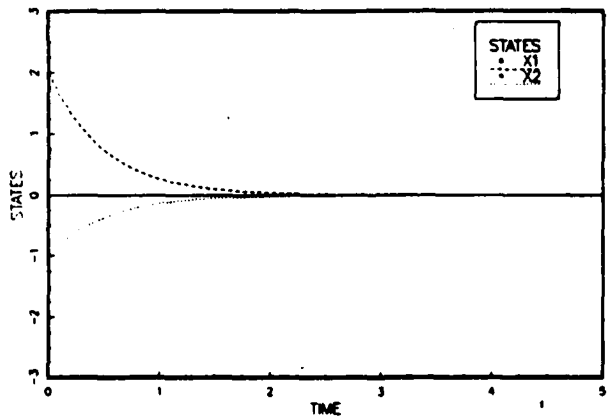


Figure 6. Time Response of Example 1 with Input Command u = 0.

As the graph shows, such a method takes over 3 s to bring the states close to zero. Hence, the bang-bang technique is dramatically faster than the desired state input command for a closed-loop system.

EXAMPLE 2. OPEN-LOOP SYSTEM

As stated, the open-loop poles are at -0.382 and -2.618. For the open-loop problem, we break the loop on the system and input the control directly to the plant until the state is brought to zero (or nearly to zero), and then the loop is closed again.

Since the same input, $u = \pm 1$, will be used, we should expect the bang-bang performance of the open loop to be slower than the closed loop because the dominant pole is closer to the origin in the s-plane. This is, in fact, the result which we find. The time-optimal control profile is

$$
\begin{aligned}
u &= 1 \qquad 0 \le t \le 0.916 \\
u &= -1 \qquad 0.916 \le t \le 1.233
\end{aligned}
\tag{26}
$$

Now the total time required is 1.233 s as opposed to 1.098 s. The time response appears below in Fig. 7:

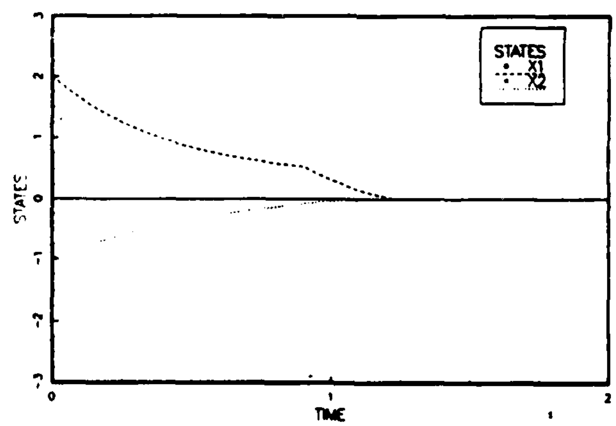
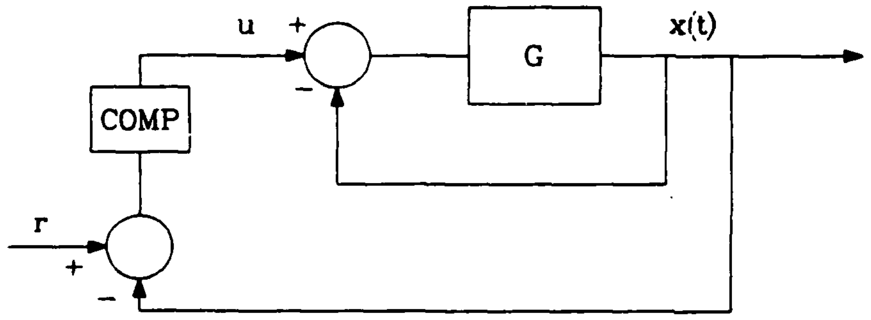


Figure 7. Time Response of Example 2.

The time required to reach the origin if the input command u = 0 is used requires over 30 s, which, of course, is much greater than the bang-bang requirement.

EXAMPLE 3. A MORE REALISTIC CLOSED-LOOP SYSTEM

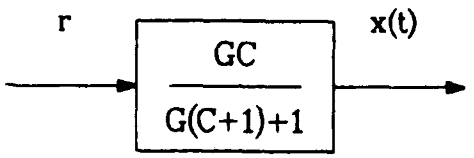
In a true closed-loop system, one does not directly input a torque to the plant and attempt to use that as a control, but more often will input a reference command, which acts on some compensation to provide a torque which will drive the system. The diagram below shows this configuration, which resembles compensation on existing BSMs (Ref. 5).



$$G = \frac{1}{(s+p_1)(s+p_2)}$$

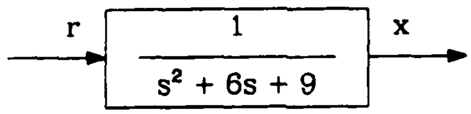
$$COMP = \frac{s+a}{s+b}$$

This can be simplified to



$$\text{Transfer Function} = \frac{s+a}{(s+p_1)(s+p_2)(s+b) + 2s+a+b}$$

For the open-loop system of example 2 and a compensator C = s+1/s+4. we have



$$\dot{\bar{x}} = \begin{bmatrix} -6 & -9 \\ 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

Now we can do this problem for the compensated system. However, we must determine limits for r. We know that $|u| \leq 1$. Also, at low frequencies the compensator gain is a/b or 1/4. Assuming small departures from the origin, then $|r| \leq 4$. We can now solve the bang-bang problem for $r = \pm 4$.

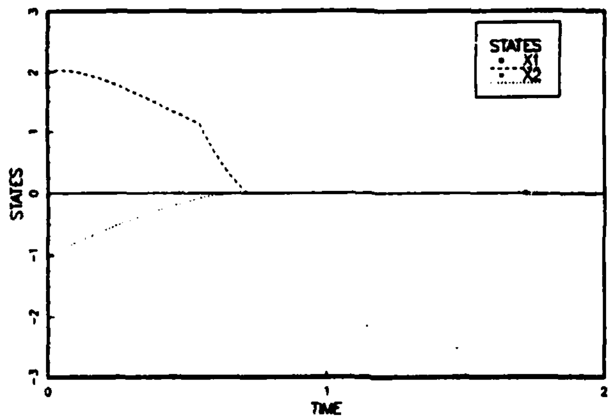
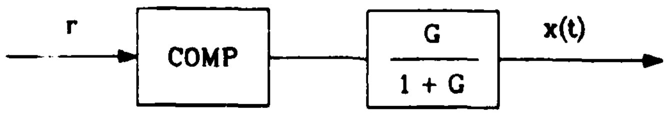


Figure 8. Time Response of Example 3.

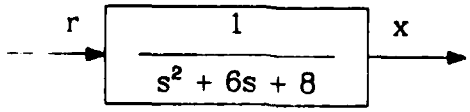
Figure 8 shows the time response for this example. The switch from u = 4 to u = -4 is made at approximately 0.548 s and the state reaches the origin after a total of 0.718 s, which is a shorter time than required by the uncompensated closed loop. We should expect this because the effect of the compensation is to drive the s-plane poles to faster locations.

EXAMPLE 4. A REALISTIC OPEN-LOOP SYSTEM

The system of Example 3 may be controlled in an open loop. The system would be represented as



or in block-diagram form



which in state-space form is

$$\dot{\bar{x}} = \begin{bmatrix} -6 & -8 \\ 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \qquad (27)$$

Again, the input used is $r = \pm 4$. The result, shown in Fig. 9 is that the switch occurs at 0.585 s and takes a total time of 0.746 s to reach the origin.

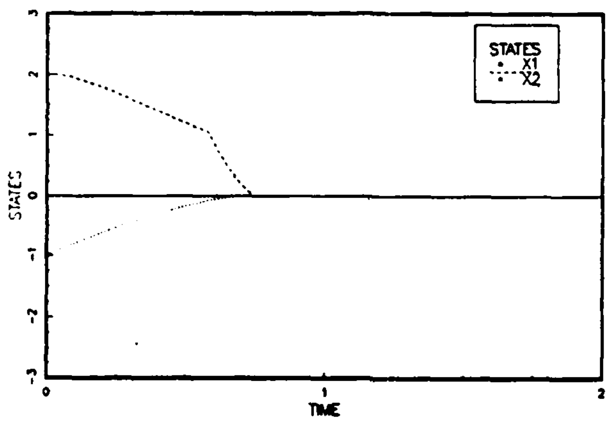


Figure 9. Time Response of Example 4.

As expected, this result is just slightly slower than the closed loop, but is much faster than the uncompensated open-loop and closed-loop examples.

REAL WORLD CONSIDERATIONS

The examples shown are, of course, much simpler than the systems that are encountered when considering actual BSMs. The system dynamics for a BSM alone can be modeled as a fourth-order or sixth-order system. Accounting for the entire control system can make the model a tenth or higher order problem. Hence, the computation involved for the method of Examples 3 and 4 can be prohibitive, particularly in a system where the computation is real-time. Additionally, the closed-loop will invoke more orders if the feedback loop is something other than mere proportional control. However, the closed-loop approach in general should not be much more complicated than the open-loop in terms of computation.

CONCLUSION

Closed-loop bang-bang control provides a faster way of retargeting systems than does open-loop control. Although only simple real-pole examples have been given here, the results extend to the higher-order complex-pole case. Closed-loop control takes advantage of inherently faster poles at a low cost in increased complexity.

IV. SOLUTION OF THE BANG-BANG BOUNDARY VALUE PROBLEM FOR Nth-ORDER SYSTEMS WITH COMPLEX EIGENVALUES

MOTIVATION

Existing solutions for the bang-bang control problem tend toward either solutions to the real eigenvalue problem (Ref. 6) or very broad approaches that produce bang-bang solutions by a limiting process (Ref. 7). In this section, a specific solution will be demonstrated that assumes the bang-bang profile for the optimal control given the maximum amplitudes of the control variable, a postulated number of switchings, and an initial guess at the control profile.

The bang-bang boundary value problem attempts to take the states of a system from an initial condition in a desired final condition in the shortest possible time. If a solution exists, it uses only the maximum and minimum values of u, switching between the two with a frequency and duration that is time-optimal. For a system of real eigenvalues, it is well-known that the optimum number of switchings is n-1, where n is the order of the system (Ref. 8). For a system of complex eigenvalues, no such guarantee can be made, but often n-1 is the optimum number of switchings. For a second-order system, the optimum bang-bang profile can be found using switching curves (Ref. 9). For a problem which requires more than one switching, switching curve analysis is inadequate, and the control profile must be found computationally. This analysis can be done by uncoupling the system and using the boundary conditions to fashion a series of complex nonlinear equations which, when solved simultaneously, yield the optimum control. To decouple the system, we use a similarity transformation as follows:

A system whose dynamics and control are defined by

$$\dot{\bar{x}} = A\bar{x} + \bar{b}u \tag{28}$$

where $\bar{x}$ is the state vector of dimension n, A is an n x n dynamics matrix, $\bar{b}$ is an n vector and u is a scalar, may be transformed by a similarity

transformation $\bar{y} = P^{-1}\bar{x}$, where P is a modal matrix of the column eigenvectors of A. Such a transformation yields $P^{-1}A\,P = \Lambda$, where Λ, where Λ is the diagonal matrix of the eigenvalues of A. We then have

$$\dot{\bar{y}} = \Lambda\bar{y} + P^{-1}\,\bar{b}u \tag{29}$$

In a physical system, some of the eigenvalues are complex conjugate pairs and the remaining are real poles. Hence, for a system of m complex conjugate pairs, there are n-2m real poles. Because the dynamics matrix is diagonal in y-space, the form of solution for each component of $\bar{y}$ is a simple exponential with a constant particular solution, since u is constant for the bang-bang problem. However, the solutions for states with complex eigenvalues are complex and, thus, yield two equations, one each for the real and imaginary parts. A transformation can be chosen which results in duplication between the conjugate states so that each complex pair yields only two unique equations. The result is that a boundary value problem for an n-order system of complex poles may be characterized by n equations in time, and if a bang-bang solution exists, it may be found exactly using these equations.

SOLUTION

From Equation 29, each state in y-space is described by the equation (using component notation)

$$\dot{y}_A = (\sigma_A + \imath\omega_A)y_A + P_{Aj}^{-1}b_j u \tag{30}$$

where $\sigma_A + \imath\omega_A$ is the Ath eigenvalue of the system, and this has the solution

$$y_A(t) = \left(y_{A0} + \frac{P_{Aj}^{-1}b_j u}{\sigma_A + \imath\omega_A}\right) e^{(\sigma_A + \imath\omega_A)t} - \frac{P_{Aj}^{-1}b_j u}{\sigma_A + \imath\omega_A} \tag{31}$$

Now for the boundary-value bang-bang problem, initial and final conditions have been fixed, an unknown elapsed time, (which should be minimized) and an unknown number of switchings. Experience shows that it is best to guess that there will be n-1 switchings, although the number of switchings cannot be guaranteed for the bang-bang problem with complex poles (Ref. 7). A minimum

total time may be guessed by solving a suboptimal bang-bang problem, i.e., solving the single-switch bang-bang problem for the two most important states. Once these guesses have been made, we are ready for the first try at solving the time-optimal control profile.

IMPLEMENTATION

The optimal control is the solution to a set of n simultaneous nonlinear equations. The problem is more easily solved when it is reformulated as follows. Define k as the number of steps in the control profile, so we have for the state A at step k

$$y_{kA} = \left(y_{(k-1)A} + \frac{P_{Aj}^{-1} b_j u_k}{\sigma_A + \imath\omega_A} \right) e^{(\sigma_A + \imath\omega_A)t_k} - \frac{P_{Aj}^{-1} b_j u_k}{\sigma_A + \imath\omega_A} \tag{32}$$

This formulation can be further simplified (in the manner of Ref. 6) by the following identities:

$$eig_A = \sigma_A + \imath\omega_A \qquad\qquad \alpha_{kA} = \frac{P_{Aj}^{-1} b_j u_k}{eig_A}$$

and $x_k = e^{-t_k}$ (where t_k is the optimum length of time to impose the control value u_k). Then

$$y_{kA} = (y_{(k-1)A} + \alpha_{kA}) x_k^{-eig_A} - \alpha_{kA} \tag{33}$$

If the initial condition is y_{0A} and our final condition is y_{mA}, where m is the total number of steps, then a simple calculation shows that

$$y_{mA} = [y_{0A} + \alpha_{1A}](x_1 x_2 \ldots x_m)^{-eig_A} + [\alpha_{2A} - \alpha_{1A}](x_2 x_3 \ldots x_m)^{-eig_A}$$
$$+ \bullet \bullet \bullet + [\alpha_{mA} - \alpha_{(m-1)A}] x_m^{-eig_A} - \alpha_{mA} \tag{34}$$

With further simplification $\quad w_k = \prod_{j=k}^{m} x_j \tag{35}$

$$y_{mA} = [y_{0A} + \alpha_{1A}]w_1^{-eig_A} + \sum_{k=1}^{m-1} \left\{[\alpha_{(k+1)A} - \alpha_{kA}]w_{k+1}^{-eig_A}\right\} - \alpha_{mA} \tag{36}$$

and by defining $c_{1A} = y_{0A} + \alpha_{1A}$ and $c_{iA} = \alpha_{iA} - \alpha_{(i-1)A}$ for $i = 2 \ldots m$, we have

$$y_{mA} = \sum_{k-1}^{m} c_{kA} w_k^{-eig_A} - \alpha_{mA} \tag{37}$$

For each state this gives a nonlinear equation for the final state as a function of the m w_ks. We must solve the equations of all n states for w_k simultaneously. To do this, we make a first guess at the optimal profile by specifying initial values w_k for $k = 1.2 \ldots m$ in accordance with the definition (Equation 35), and find a result for y_{mA}, which can then be compared to the target result $y^{\dagger}_{mA}$.

By this comparison, we can estimate the necessary change in w_k by invoking this simple linearization:

$$y^{\dagger}_{mA} \sim y_{mA} + \frac{\partial y_{mA}}{\partial w_1} \delta w_1 + \frac{\partial y_{mA}}{\partial w_2} \delta w_2 + \ldots + \frac{\partial y_{mA}}{\partial w_m} \delta w_m \tag{38}$$

To solve for the partial derivatives, it is necessary to look again at the form of y_{mA}:

$$Y_{mA} = \sum_{k=1}^{m} c_{kA} w_k^{-eig_A} - \alpha_{mA} \tag{39}$$

For $\frac{\partial y_{mA}}{\partial w_k}$, then, we must address only the term $c_{ka} w_k^{-eig_A}$. But, for a state A which has a complex eigenvalue, c_{kA} and eig_A are complex expressions and may be written

$$c_{kA} = \beta_{kA} + \imath\gamma_{kA} \qquad eig_A = \sigma_A = \imath\omega_A \tag{40}$$

So now the term is

$$(\beta_{kA} + \iota\gamma_{kA})w_k^{-\sigma_A - \iota\omega_A}$$

$$= w_k^{-\sigma_A}[\beta_{kA}\cos\,\omega_A \ln w_k + \gamma_{kA}\,\sin\,\omega_A \ln w_k$$

$$+ \iota(\gamma_{kA}\cos\,\omega_A \ln w_k - \beta_{kA}\sin\,\omega_A \ln w_k)] \quad (41)$$

It is most efficient to analyze the real and imaginary parts of this term separately. This can be done because, for the similarity transformation chosen, each state of a complex eigenvalue has a complex conjugate state. Therefore, we define the real part and imaginary part of a complex state as separate states, ignore the conjugate state and thus retain the same number of states. Define $\tilde{y}_{kA}$ as the real part of y_{kA} and y^*_{kA} as the imaginary part of y_{kA}. Then

$$\frac{\partial \tilde{y}_{mA}}{\partial w_k} = -\sigma_A w_k^{-\sigma_A - 1}[\beta_{kA}\cos\,\omega_A \ln w_k + \gamma_{kA}\sin\,\omega_A \ln w_k]$$

$$+ \omega_A w_k^{-\sigma_A - 1}[\gamma_{kA}\cos\,\omega_A \ln w_k - \beta_{kA}\sin\,\omega_A \ln w_k] \quad (42)$$

$$\frac{\partial y^*_{mA}}{\partial w_k} = -\sigma_A w_k^{-\sigma_A - 1}[\gamma_{kA}\cos\,\omega_A \ln w_k - \beta_{kA}\sin\,\omega_A \ln w_k]$$

$$- \omega_A w_k^{-\sigma_A - 1}[\gamma_{kA}\sin\,\omega_A \ln w_k + \beta_{kA}\cos\,\omega_A \ln w_k] \quad (43)$$

Of course, for the state of a real eigenvalue the equations are much simpler and there need be no auxiliary states:

$$\frac{\partial y_{mA}}{\partial w_k} = -eig_A c_{kA} w_k^{-eig_A - 1} \quad (44)$$

Now we have an expression for each first partial with respect to w_k for each unique state and we have

$$[\Delta]\delta\bar{w} = \bar{y}^{\dagger} - \bar{y}_m \quad (45)$$

where Δ =

$$\begin{bmatrix} \frac{\partial \tilde{y}_{m1}}{\partial w_1} & \frac{\partial \tilde{y}_{m1}}{\partial w_2} & \cdots & \frac{\partial \tilde{y}_{m1}}{\partial w_m} \\ \frac{\partial y^{*}_{m1}}{\partial w_1} & \frac{\partial y^{*}_{m1}}{\partial w_2} & \cdots & \frac{\partial y^{*}_{m1}}{\partial w_m} \\ \frac{\partial \tilde{y}_{m3}}{\partial w_1} & \frac{\partial \tilde{y}_{m3}}{\partial w_2} & \cdots & \frac{\partial \tilde{y}_{m3}}{\partial w_m} \\ \vdots & & & \\ \frac{\partial y_{mn}}{\partial w_1} & \cdots & & \frac{\partial y_{mn}}{\partial w_m} \end{bmatrix} \qquad (46)$$

Using a linear solver we can solve a first-order approximation of $\delta\bar{w}$ and use that to adjust our guess at $\bar{w}$. It is clear from the definition of w_k that $0 < w_1 < w_2 < \ldots < w_m < 1$. If $\delta\bar{w}$ causes the new guess to violate this identity, then a first order approximation is not good enough. For a complex state, the expressions for

$$\frac{\partial^2 \tilde{y}_{mA}}{\partial w_k^2} \text{ and } \frac{\partial^2 y^{*}_{mA}}{\partial w_k^2}$$

are easily calculated from Equations 43 and 44. For a real eigenvalue state, the higher order derivatives follow from 45.

A second-order approximation may be used as follows. The governing equation is

$$y^{\dagger}_{mk} \sim y_{mk} + \sum_{i=1}^{m} \frac{\partial Y_{mA}}{\partial w_i} \delta w_i + \sum_{i=1}^{m} \frac{\partial^2 y_{mk}}{\partial w_i^2} (\delta w_i)^2 \qquad (47)$$

A linear solver cannot be used with this equation. However, using the results from a first-order guess at $\delta\bar{w}$, as detailed, we can reformulate the equation:

$$y^{\dagger}_{A} \approx y_{mA} + \sum_{i=1}^{m} \left[\frac{\partial y_{mA}}{\partial w_i} + \frac{\partial^2 y_{mA}}{\partial w_i^{\,2}} \, \delta w_i^{*} \right] \delta w_i \tag{48}$$

where δw_i^{*} is the result of a first-order solution for δw_i. Now we can use a linear solver to find $\delta\bar{w}$. Fitting this solution back into the equation, we can reiterate until we get a reasonable estimate for $\delta\bar{w}$. Using that, we can change $\bar{w}$, and iterate using the first-order (and higher-order if desired) approximations for $\delta\bar{w}$ until an exact solution is found.

EXAMPLE

Consider the following system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with control $-1 \leq u \leq 1$. This represents a third-order system with poles $\lambda = -2, -1 \pm i$. Let us define a boundary value problem in which the initial state, x_0, is $[1\ 0\ 0]^T$ and the final state, x_f, is $[0\ 0\ 0]^T$. A convenient transformation is

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -2 & i-1 & -1-i \\ 4 & -2i & 2i \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix}$$

Note that the column vectors of P corresponding to the complex conjugate states are themselves complex conjugates. This allows us to ignore one of the states in y-space, because the corresponding states (y_2 and y_3) are complex conjugates for all time:

$$\dot{y}_1 = -2y_1 + 0.5u \qquad y_0 = [1 \ -\imath \quad \imath]^T$$

$$\dot{y}_2 = (-1+\imath)y_2 - 0.25(\imath+1)u \qquad y_f = [0 \quad 0 \quad 0]^T$$

$$y_3 = (-1-\imath)y_3 + 0.25(\imath-1)u \tag{49}$$

Ignore y_2; do the bang-bang problem, define u = 1 and select the number of switchings to be two, since that is one less than the order of the system. Clearly, the only sequence of controls that will work for two switchings is $u_1=-1$, $u_2=1$, $u_3=-1$. From this and the definition for α, it follows that $\alpha_{11} = 0.25$, $\alpha_{21} = -0.25$, $\alpha_{31} = 0.25$, $\alpha_{13} = \imath/4$, $\alpha_{23} = -\imath/4$, and $\alpha_{33} = \imath/4$. Using (8) we have

$$y_{31} = (5/4)w_1^{\,2} - (1/2)w_2^{\,2} + (1/2)w_3^{\,2} - 1/4$$

$$y_{33} = (5/4)\imath w_1^{\,1+\imath} - (\imath/2)w_2^{\,1+\imath} + (\imath/2)w_3^{\,1+\imath} - \imath/4 \tag{50}$$

Or, separating the real from the imaginary

$$\tilde{y}_{33} = -(5/4)w_1 \sin(\ln w_1) + (1/2)w_2 \sin(\ln w_2)$$
$$-(1/2)w_3 \sin(\ln w_3)$$

$$y^*_{33} = (5/4)w_1 \cos(\ln w_1) - (1/2)w_2 \cos(\ln w_2)$$
$$+ (1/2)w_3 \cos(\ln w_3) - 1/4 \tag{51}$$

The first-order partials fall out directly:

$$\frac{\partial \tilde{y}_{33}}{\partial w_1} = -5/4[\sin(\ln w_1) + \cos(\ln w_1)]$$

$$\frac{\partial y^*_{33}}{\partial w_1} = 5/4[\cos(\ln w_1) - \sin(\ln w_1)]$$

$$\frac{\partial y_{31}}{\partial w_1} = 5/2w_1 \tag{52}$$

The derivatives for the other w_is are similar. We are now ready to guess at the control profile and begin solving for it. A simple switching curve analysis between x_1 and x_2 shows that the total time for the problem may be around 1 s. Using the definition for w, we can guess $w_1 = 0.4$, $w_2 = 0.6$ and $w_3 = 0.7$. Using these values, we get for the final states: $y_{31} = 0.015$, $\tilde{y}_{33} = 0.3722$ and $y^*_{33} = 0.1207$ where the final states are $y^\dagger_{31} = 0$, $\tilde{y}^\dagger_{33} = 0$ and $y^{*\dagger}_{33} = 0$.

Now, we can use Equation 47 to find a correction to $\bar{w}$. We have

$$\Delta^{\sim} \begin{bmatrix} 1.0000 & -0.6000 & 0.7000 \\ 0.2307 & 0.1917 & -0.2940 \\ 1.7526 & -0.6806 & 0.6431 \end{bmatrix} \text{ so that } [\Delta]\delta\bar{w} = \begin{bmatrix} -0.0150 \\ -0.3722 \\ -0.1207 \end{bmatrix} \tag{53}$$

Hence, $\delta\bar{w} = [-1.2528\ -7.2490\ -4.451]$. These are values that have no physical meaning, so we must go to a higher order calculation. We can now use Equation 48, where $\delta\bar{w}^*$ is the result of the first-order calculation. Equations for the second-order partials follow from Equation 53. Now we begin a cycle of iterations on $\delta\bar{w}$ until we converge on a value that will satisfy the bounds on $\bar{w}$. In this case, we arbitrarily chose to iterate 30 times to arrive at $\delta\bar{w} = [-\ 0.3286\ -\ 0.4819\ -\ 0.0167]^T$ As a result, $\bar{w} = [0.0714\ 0.1181\ 0.6833]^T$. With this value for $\bar{w}$, the final states are now $y_{31} = -\ 0.017$, $\tilde{y}_{33} = 0.1201$ and $y^*_{33} = 0.0206$. These are much closer to the target end condition than the first guess. We are now close enough to the desired states to use the first-order method to get to a final solution. With only five iterations we get to within 10^{-7} of the target for all three states. The final $\bar{w}$ is $[0.0619\ 0.2766\ 0.7529]^T$, which mandates a control time profile of $t_1 = 1.4963$, $t_2 = 1.0015$ and $t_3 = 0.2838$. This time-optimal control profile is shown in Fig. 10.

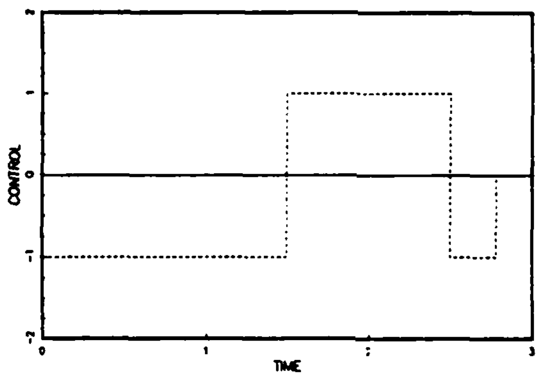


Figure 10. Time-optimal control profile.

The phase plots for x_1 versus x_2 (Fig. 11) and x_2 versus x_3 (Fig. 12) show that this solution brings all the states to the desired condition simultaneously.

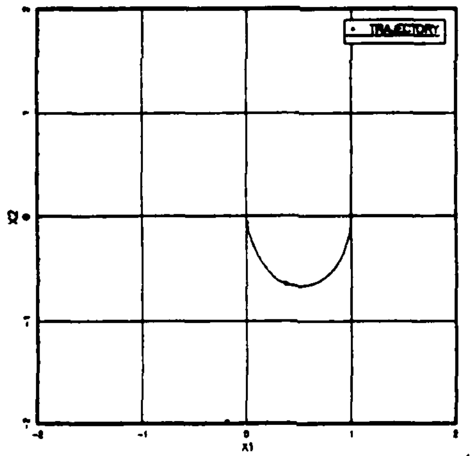


Figure 11. X_1 versus X_2.

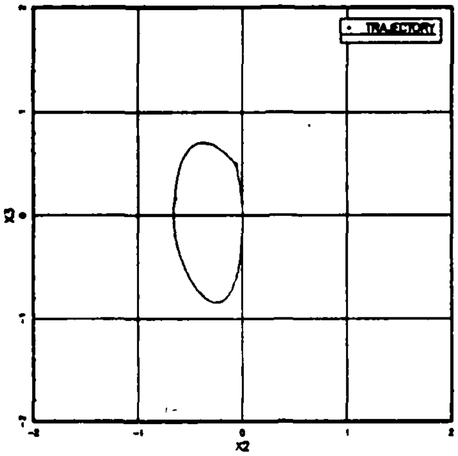


Figure 12. X_2 versus X_3.

V. OPTICAL RETARGETING PERFORMANCE ENHANCEMENTS DUE TO TIME-OPTIMAL STEERING FOR TYPE II SYSTEMS

INTRODUCTION

In this section, a simple type II control loop is used to derive general conclusions about the retargeting time performance enhancement rendered by time-optimal control.

MODELING

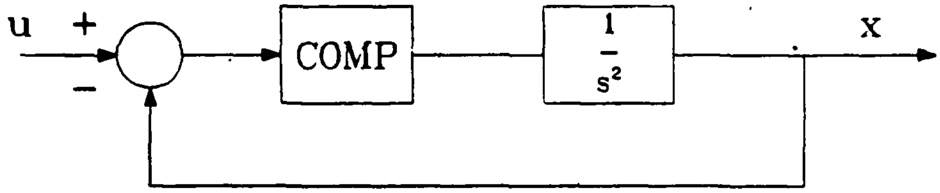
For this analysis, a simple lead compensator of bandwidth ω_0 and lead ratio α on a pure inertia in a unity gain feedback is used. (See Fig. 13.)

For this system the transfer function is:

$$\frac{x}{u} = \frac{\omega_0^2\sqrt{\alpha}(s + \omega_0/\sqrt{\alpha})}{s^3 + \omega_0\sqrt{\alpha}\, s^2 + \omega_0^2\sqrt{\alpha}\, s + \omega_0^3} \tag{54}$$

SYS, the code used in this study to find time-optimal switchings, uses a state-space system representation $\dot{\bar{x}} = A\bar{x} + \bar{b}u$, where $\bar{x}$ is the state vector. A is the dynamics matrix, $\bar{b}$ is the input vector and u is a scalar command (Ref. 10). We will address only the single-input single-output problem. The system of Fig. 13 can be put in the following canonical time-domain form

$$\dot{\bar{x}} = \begin{bmatrix} -\omega_0\sqrt{\alpha} & 1 & 0 \\ -\omega_0^2\sqrt{\alpha} & 0 & 1 \\ -\omega_0^3 & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ \omega_0^2\sqrt{\alpha} \\ \omega_0^3 \end{bmatrix} u \tag{55}$$



$$COMP = \omega_0^2\sqrt{\alpha}\ \frac{s+\omega_0/\sqrt{\alpha}}{s+\omega_0\sqrt{\alpha}}$$

Figure 13. Type II system with lead compensation.

where x is the element of the state vector $\bar{x}$ that corresponds to position. For most practical cases, however, the A matrix will be highly ill-conditioned. Because SYS requires a well-conditioned dynamics matrix, it is necessary for us to scale A.

SCALING

Time-scaling

The easiest scaling is time-scaling. If we change the problem from time t to a normalized time τ, by a factor γ, we have

$$\tau = \frac{t}{\gamma} \quad d\tau = \frac{dt}{\gamma} \quad \frac{d\bar{x}}{d\tau} = \gamma\frac{d\bar{x}}{dt},$$

so

$$\frac{d\bar{x}}{d\tau} = \gamma A\bar{x} + \gamma\bar{b}u \tag{56}$$

This scaling will not change the condition number of the matrix, since all the elements are simply multiplied by the scale factor γ, but it can make the solution of the time-optimal problem easier. Because SYS weights time in exponential functions in its solution, it works best for solutions on the order of t = 1.

Magnitude-scaling

The only type of scaling that will improve the condition number of A is scaling of the separate states. The canonical form chosen to represent the system makes this scaling easy. Shown in Equation 57 is a time and magnitude scaled system representation, where β and λ are magnitude scale factors and γ is the time scale factor.

$$\dot{\bar{x}} = \begin{bmatrix} \omega_0\sqrt{\alpha}\gamma & \beta\gamma & 0 \\ \dfrac{-\omega_0^2\sqrt{\alpha}\gamma}{\beta} & 0 & \dfrac{\lambda\gamma}{\beta} \\ \dfrac{-\omega_0^3\gamma}{\lambda} & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ \dfrac{\omega_0^2\sqrt{\alpha}\gamma}{\beta} \\ \dfrac{\omega_0^3\gamma}{\lambda} \end{bmatrix} u \tag{57}$$

Now taking a general scaling approach in which $\beta = \omega_0$, $\lambda = \omega_0^2$ and $\gamma = \frac{1}{\omega_0}$, renders

$$\dot{\bar{x}} = \begin{bmatrix} -\sqrt{\alpha} & 1 & 0 \\ -\sqrt{\alpha} & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ \sqrt{\alpha} \\ 1 \end{bmatrix} u \tag{58}$$

This scaling shows that the system response is not a function of bandwidth, but only of the lead ratio α. Bandwidth is reciprocally related to the time scaling. For example, a simple type II system with a bandwidth of 1 Hz has the exact response of a 10 Hz system on a time scale compressed by a factor of 10.

SYSTEM EIGENVALUES

Before examining the effects of a time-optimal solution to our third-order system, we should look at its eigenvalues. Their value is a function of α:

for $\alpha < 9$

$$\lambda_1 = -\omega_0$$

$$\lambda_{2,3} = -\frac{\omega_0}{2}\left[(\sqrt{\alpha}-1) \pm \imath\sqrt{3 + 2\sqrt{\alpha} - \alpha}\right]$$

for $\alpha = 9$

$$\lambda_{1,2,3} = -\omega_0$$

and

for $\alpha > 9$

$$\lambda_1 = -\omega_0$$

$$\lambda_{2,3} = -\frac{\omega_0}{2}\left[(\sqrt{\alpha}-1) \pm \sqrt{\alpha-2\sqrt{\alpha}-3}\right] \tag{59}$$

We can also easily solve the time-domain equations for the state x_1:

for $\alpha < 9$

$$x_1 = c_1 e^{-\omega_0 t} + e^{-\omega_0(\sqrt{\alpha}-1)t/2}\left[c_2 \cos\left(\frac{\sqrt{3 + 2\sqrt{\alpha}-\alpha}}{2}\omega_0 t\right) + c_3 \sin\left(\frac{\sqrt{3 + 2\sqrt{\alpha}-\alpha}}{2}\omega_0 t\right)\right] + u \tag{60}$$

for $\alpha = 9$

$$x_1 = e^{-\omega_0 t}\left[c_1 + c_2 t + c_3 t^2\right] + u \tag{61}$$

and for $\alpha > 9$

$$x_1 = c_1 e^{-\omega_0 t} + c_2 e^{-\omega_0/2[(\sqrt{\alpha}-1) + \sqrt{\alpha-2\sqrt{\alpha}-3}]t} + c_3 e^{-\omega_0/2[(\sqrt{\alpha}-1) - \sqrt{\alpha-2\sqrt{\alpha}-3}]t} + u \tag{62}$$

x_2 and x_3 follow immediately from the state equations and the coefficients c_1, c_2, and c_3 are functions of $\sqrt{\alpha}$ and u. What we can see from this is that the poorest performance in time for a classical set-point change comes from systems whose α is farthest from 9. This is true because as α moves away from 9, the real part of one of the poles (or the complex pole pair) moves toward the origin creating a slower response. We will find that the

time-optimal method is, for the most part, relatively insensitive to this trend, and, hence, the time-optimal advantage over the classical set-point change reaches a lower limit at $\alpha = 9$, when the eigenvalues are triply redundant at ω_0.

TIME-OPTIMAL SOLUTION

The time-optimal solution for a system like this, in which the control, u, is constrained in magnitude, is a switching between the maximum and minimum values for u (Ref. 12). For a system of real eigenvalues, the number of switchings is n-1, where n is the order of the system (Ref. 13). Even for a system with complex eigenvalues, my experience shows that small displacements from zero initial conditions, a type of problem consistent with beam steering, also require only n-1 switchings. Therefore, in order to solve the time-optimal solution for such a problem, we assume a solution in which u switches between its maximum and minimum value (the same maximum magnitude with different signs) n-1 times. The solution is found by determining the exact time lengths for each leg of the control profile that bring the system from the initial condition to a target condition. For instance, for $\alpha = 10$ we have

$$\dot{\bar{x}} = \begin{bmatrix} -3.16228 & 1 & 0 \\ -3.16228 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 3.16228 \\ 1 \end{bmatrix} u \qquad (63)$$

From this we can define the problem

$$|u| \leq \Phi \quad \bar{x}_0 = [0\ 0\ 0]^T \text{ and } \bar{x}^{\dagger} = [1 \quad 0.31623 \quad 0] \qquad (64)$$

where $\bar{x}_0$ is a zero initial condition and $\bar{x}^{\dagger}$ is a steady-state end condition that will hold $x_1 = 1$. The magnitude limit of u is described by Φ. To take advantage of the time optimal technique, u must be larger in magnitude than what is required to reach $\bar{x}^{\dagger}$ using a classical set-point change. For instance,

if we were to define the u-limit as $\Phi = 4$, then we would find that the system will move much more quickly switching between $u = \pm 4$ than it will with just inputing $u = 1$. The gain in retargeting performance by using time-optimal switching for this problem is a function of the ratio of Φ to the magnitude of the classical u input. We will call this ratio the overtorque ratio. This functional dependence will be investigated parametrically against α in the next paragraph.

TIME-OPTIMAL RETARGETING PERFORMANCE AS A FUNCTION OF THE OVERTORQUE RATIO

The results of time-optimal simulations will be examined and compared to the classical set-point change. The standard for comparison will be the time it takes to get the system from $\bar{x}_0$ to $\bar{x}^+$ within a tolerance of 5 percent (i.e., state x_1 must be within 5 percent of $\bar{x}^+_1$).

Three cases will be examined in detail: $\alpha = 4$, $\alpha = 16$ and $\alpha = 9$. In each case, the time to final state was determined for overtorque ratios ranging from 1.01 to 100. Comparing these times against the time it takes for a classical set-point change, we can derive effective retargeting bandwidth (ERB) ratios. For instance, in the case of $\alpha = 9$, an overtorque ratio of 2 results in a time of 3.05 time units from $\bar{x}_0$ to $\bar{x}^+$ (time units are normalized on a bandwidth of $\omega_0 = 1$). The classical set-point change requires 6.57 time units. Thus, the time-optimal profile enhances the retargeting performance by a factor of 2.15. It gives the system the retargeting response of a system with bandwidth $\omega_0 = 2.15$. We then say that the ERB ratio is 2.15.

In Table 1, some of the results of the time-optimal simulations have been tabulated. A number of things can be seen from this chart. The most obvious conclusion is that even for overtorque ratios close to 1, you get a considerable reduction in retargeting time. Also, it can be seen that the time-optimal performance is relatively insensitive to variations in α, particularly for large overtorque rations. Figure 14 shows a graph of the ERB ratios versus the overtorque ratios.

TABLE 1. Time-Optimal Solutions for 5 percent settling criterion.

	α = 4		α = 16		α = 9	
Normalized Classical Set-point Change Time	7.36		7.78		6.57	
Overtorque Ratio	Normalized Retargeting Time	ERB Ratio	Normalized Retargeting Time	ERB Ratio	Normalized Retargeting Time	ERB Ratio
1.1	3.88	1.90	7.98	0.97	5.59	1.18
2	2.63	2.80	3.60	2.16	3.05	2.15
5	1.82	4.04	2.11	3.69	1.94	3.39
10	1.43	5.15	1.56	4.99	1.49	4.41
20	1.13	6.51	1.20	6.48	1.16	5.56
50	0.83	8.87	0.86	9.05	0.85	7.73
100	0.66	11.15	0.68	11.44	0.68	9.66

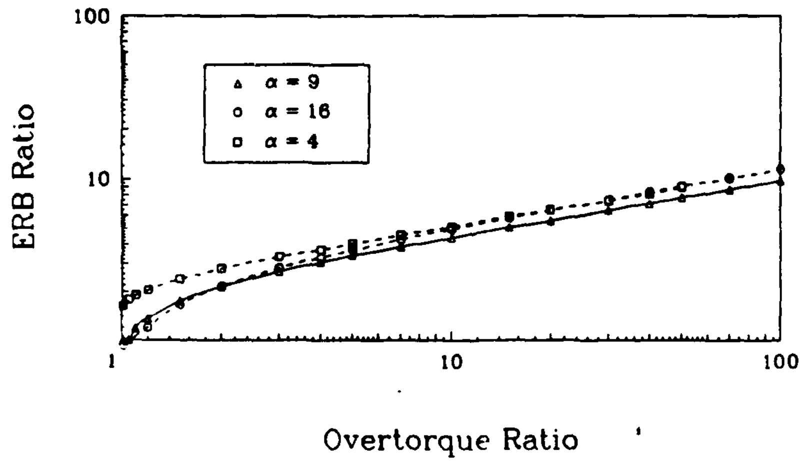


Figure 14. ERB Ratio versus overtorque ratio for 5 percent settling criterion.

The real benefit of time-optimal control is not seen until accuracy requirements become more rigorous. As we move from a 5 percent settling criterion to a perfect settling criterion, the time needed for a classical set-point change becomes much longer, while the time-optimal solution does not change significantly. Table 2 shows the tabulated results for the near

perfect settling case in which the settling criterion is 1/1000 percent. Figure 15 shows the ERB ratio-overtorque ratio graph for the perfect case. Here α = 9 is the limiting case for time-optimal enhancement. Note that the knee of the curve is much closer to 1 than it is for the 5 percent settling requirement. This demonstrates that as settling requirements become more rigorous, the time-optimal control renders a greater savings in time for lower overtorque ratios.

TABLE 2. Time-Optimal Solutions for perfect settling criterion.

	α = 4		α = 16		α = 9	
Normalized Classical Set-point Change Time	25.53		34.18		18.21	
Overtorque Ratio	Normalized Retargeting Time	ERB Ratio	Normalized Retargeting Time	ERB Ratio	Normalized Retargeting Time	ERB Ratio
1.01	10.08	2.53	14.18	2.41	9.04	2.01
1.1	4.49	5.69	8.37	4.08	6.05	3.01
2	2.85	8.97	3.74	9.11	3.22	5.66
5	1.93	13.26	2.18	15.65	2.03	8.97
10	1.50	17.03	1.61	21.15	1.55	11.79
20	1.18	21.64	1.23	27.65	1.20	15.16
50	0.87	29.51	0.89	38.51	0.87	20.85
100	0.69	37.24	0.70	49.03	0.69	26.41

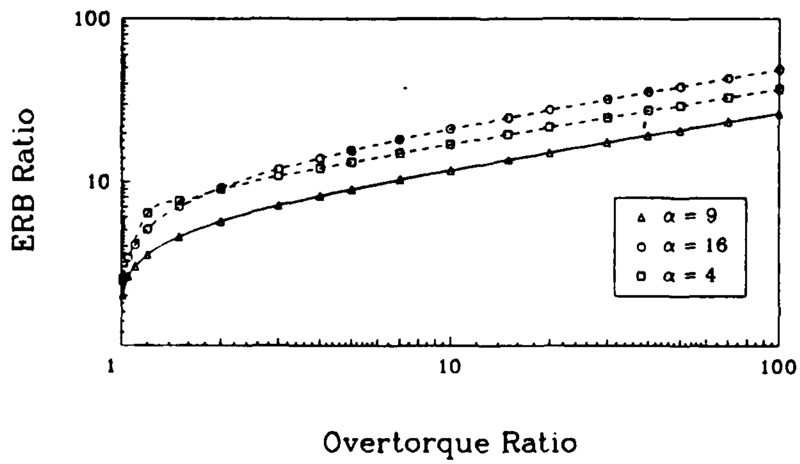


Figure 15. ERB Ratio versus overtorque ratio for perfect settling criterion.

CONCLUSION

Time-optimal methods for retargeting render a dramatic improvement in retargeting performance. This has been demonstrated using a simple Type II unity gain feedback system with a lead compensator. Even for the limiting case, time-optimal techniques provide a lucrative improvement in performance. A small ratio of overtorque results in a greatly increased speed of retargeting, and as settling requirements become more rigorous, time-optimal techniques offer even more of an advantage.

VI. SOLUTION OF THE TIME-OPTIMAL RETARGETING OF A 9TH-ORDER BSM MODEL

MODELING

Reference 14 describes the modeling of the laboratory setup. A large block diagram of both BSM axes was produced and the system in the s-domain was described. This model was taken and combined into a single block for a closed-loop representation of a single axis. The transfer function for this single axis closed loop is:

$$\frac{x}{u} = \frac{K(s^3 + A_0 s^2 + A_1 s + A_2)}{s^9 + B_1 s^8 + B_2 s^7 + B_3 s^6 + B_4 s^5 + B_5 s^4 + B_6 s^3 + B_7 s^2 + B_8 s + B_9} \quad (65)$$

where x is the output angle in radians and u is the angle command and Table 3 lists the system coefficients:

TABLE 3. System coefficients.

$B_1 = 1.037 \times 10^5$	$B_6 = 1.333 \times 10^{26}$	$A_0 = 4.037^4 \times 10$
$B_2 = 1.313 \times 10^{10}$	$B_7 = 1.837 \times 10^{29}$	$A_1 = 6.379 \times 10^8$
$B_3 = 4.710 \times 10^{15}$	$B_8 = 7.328 \times 10^{30}$	$A_2 = 9.869 \times 10^{10}$
$B_4 = 5.313 \times 10^{18}$	$B_9 = 1.086 \times 10^{33}$	
$B_5 = 3.458 \times 10^{22}$		$K = 1.077 \times 10^{33}$

We can represent this system in the following canonical form:

$$\dot{\bar{x}} = \begin{bmatrix} -B_1\gamma & \alpha_1\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{B_2\gamma}{\alpha_1} & 0 & \frac{\alpha_2}{\alpha_1}\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{B_3\gamma}{\alpha_2} & 0 & 0 & \frac{\alpha_3}{\alpha_2}\gamma & 0 & 0 & 0 & 0 & 0 \\ -\frac{B_4}{\alpha_3}\gamma & 0 & 0 & 0 & \frac{\alpha_4}{\alpha_3}\gamma & 0 & 0 & 0 & 0 \\ -\frac{B_5}{\alpha_4}\gamma & 0 & 0 & 0 & 0 & \frac{\alpha_5}{\alpha_4}\gamma & 0 & 0 & 0 \\ -\frac{B_6}{\alpha_5}\gamma & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_6}{\alpha_5}\gamma & 0 & 0 \\ -\frac{B_7}{\alpha_6}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_7}{\alpha_6}\gamma & 0 \\ -\frac{B_8}{\alpha_7}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_8}{\alpha_7}\gamma \\ -\frac{B_9}{\alpha_8}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\alpha_5} \\ \frac{A_0}{\alpha_6} \\ \frac{A_1}{\alpha_7} \\ \frac{A_2}{\alpha_8} \end{bmatrix} K\gamma P_{19} u \tag{66}$$

In this representation, γ is a time-scale factor so that the time units are in terms of $r = \frac{t}{\gamma}$, where t is time in seconds, and τ is the unit of time in which this state-space form is fashioned. Hence, in a system so scaled, a time of 1 time unit, with a scale factor $\gamma = 0.001$, means 0.001 s.

The scale factors α_1 through α_8 can be used to condition the dynamics matrix for a more efficient solution to the time-optimal problem. By simply converting from state-space back to the s-domain transfer function, it can be shown that neither scale factor changes the characteristics of the system.

REDUCTION OF ORDER

Large order systems do not converge well to a solution for the time-optimal boundary-value problem unless you know to a great degree of accuracy what the time-optimal control profile is already. This is true because, for large-order systems, the canonical form for state-space used here is sparse. Also sparse is the matrix which determines the differential rate of states with respect to control leg times. The result is that for large-order problems, the SYS algorithm is unable to make the large corrections that allow for any uncertainty of the control profile solution. The remedy is to reduce the order of the system to a more forgiving space (2^{nd} or 3^{rd} order) and to build back up to the original size on the basis of the answers determined at lower orders.

The order of the system is reduced by determining the eigenvalues and selecting the dominating ones. A discussion of how this was done for the 9^{th}-order BSM model follows.

Begin by finding the poles and zeros of the system, which follow (normalized in seconds).

TABLE 4. System poles and zeros.

Poles	Zeros
$P_6 = -3000 \pm \imath 95394$	$Z_2 = -20106 \pm \imath 15079$
$P_5 = -29999$	$Z_1 = -153.22$
$P_4 = -2520 \pm \imath\, 5757$	
$P_3 = -6191$	
$P_2 = -2437$	
$P_1 = -18.26 \pm \imath 75.79$	

Now choose the dominant poles to build the lowest-order system. Clearly the pole pair P_1 is the dominant pole pair since it is closest to the origin. Then take P_1 and Z_1 to fashion a transfer function that has similar characteristics to the 9^{th}-order model. Since the gain on the full-order model is nearly one, adjust K for all cases (including the 9th-order model) to be one. For the second-order approximation, we have:

$$\frac{x}{u} = \frac{39.67(s + 153.22)}{s^2 + 36.5139s + 6077.7} \tag{67}$$

$$\dot{\bar{x}} = \begin{bmatrix} -36.5139 & 0 \\ -6077.8 & \end{bmatrix} \bar{x} + \begin{bmatrix} 39.67 \\ 6077.8 \end{bmatrix} u \tag{68}$$

Scaling in magnitude:

$$\dot{\bar{x}} = \begin{bmatrix} -36.5139 & 100 \\ -60.778 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 39.67 \\ 60.778 \end{bmatrix} u \tag{69}$$

Scaling in time (γ = .01):

$$\dot{\bar{x}} = \begin{bmatrix} -.3652 & 1 \\ -.6078 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} .3967 \\ .6078 \end{bmatrix} u \tag{70}$$

A scale factor $\gamma = 0.01$ makes this problem easier to handle. Hence, answers will be expressed in 1/100ths of seconds. For a control value $|u| = 0.01$, and a target of $x_1 = 0.001$, the time optimal control profile is

$$\begin{aligned} t_1 &= 0.449 \qquad & u &= 0.01 \\ t_2 &= 0.350 \qquad & u &= -0.01 \end{aligned} \tag{71}$$

This answer can be used as a guess for a 3^{rd}-order system with pole P_2 combined with P_1. The 3^{rd}-order solution is then used to guess at the solution for the 5^{th}-order model (adding pole pair P_4 and zero pair Z_2). Adding order by order and using the previous solution as a guess allows the 9^{th}-order model to be built up.

SOLUTION

For the 9^{th}-order model a problem is prescribed to time-optimally move a BSM on one axis from an angular deflection of 0 to one of 1 mrad. This is the same as moving x_1 in the model from 0 to 0.001. To get a true solution requires us to find a steady-state $\bar{x}$ for $x_1 = 0.001$. This steady state is used as the end condition for the time-optimal boundary-value problem, and, of course, the $\tilde{0}$ vector for the initial condition. All that remains is specifying the control values, $|u| = 0.01$, 0.005, and 0.002, which correspond to overtorque ratios of 10, 5 and 2. The solutions (in 1/100ths of seconds) follow:

TABLE 5. Time-optimal solutions for the 9^{th}-order problem.

$\|u\| = 0.01$	$t_1 = 0.4503$	$\|u\| = 0.005$	$t_1 = 0.6739$	$\|u\| = 0.002$	$t_1 = 1.2275$
	$t_2 = 0.3749$		$t_2 = 0.4688$		$t_2 = 0.5427$
	$t_3 = 0.0391$		$t_3 = 0.0433$		$t_3 = 0.0590$
	$t_4 = 0.0210$		$t_4 = 0.0182$		$t_4 = 0.0104$
	$t_5 = 0.0242$		$t_5 = 0.0262$		$t_5 = 0.0333$
	$t_6 = 0.0083$		$t_6 = 0.0064$		$t_6 = 0.0029$
	$t_7 = 0.0020$		$t_7 = 0.0015$		$t_7 = 0.0012$
	$t_8 = 0.0012$		$t_8 = 0.0011$		$t_8 = 0.0011$
	$t_9 = 0.0011$		$t_9 = 0.0009$		$t_9 = 0.0003$
Total retarget time	0.9222		1.2404		1.8787

DISCUSSION

Now the solutions themselves raise some interesting issues. The first is that, once again, time-optimal techniques dramatically improve the retargeting performance of a BSM. Using a classical set-point change for

this boundary-value problem requires 17 hundredths of a second (to settle within 50 μrad of the end state). Overdrive ratios of 10, 5, and 2 yield shortened retargeting times by factors of 18, 14, and 9, respectively.

The time-optimal profiles offer some interesting conclusions. Figure 16 shows the progression of time-optimal solutions for $|u| = 0.01$ from the 2nd-order model through the full-order model. The waveform of the final solution is dominated by the control legs due to the lower-order solutions Table 5 shows that for all values of $|u|$, the major differences in the final solutions lie with the first legs, which correspond to the dominant eigenvalues. In fact, the higher-order control legs due to the larger eigenvalues are roughly inverse functions of the eigenvalues and are relatively insensitive to differences in $|u|$. This would lead one to believe that low-order models, that account for the dominant eigenvalues, are sufficient to yield the dramatic improvements in retargeting that time-optimal methods offer.

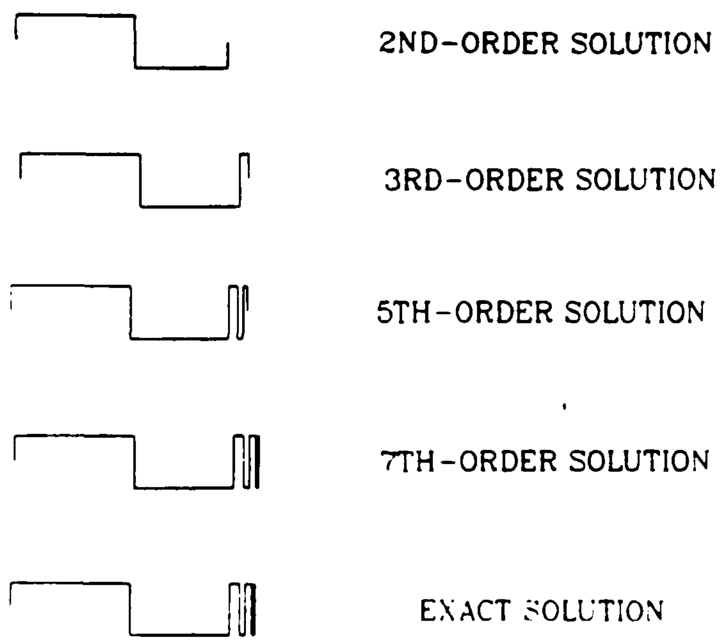


Figure 16. Progression of time-optimal solutions for $|u| = 0.01$.

PERFORMANCE OF LOW-ORDER APPROXIMATIONS

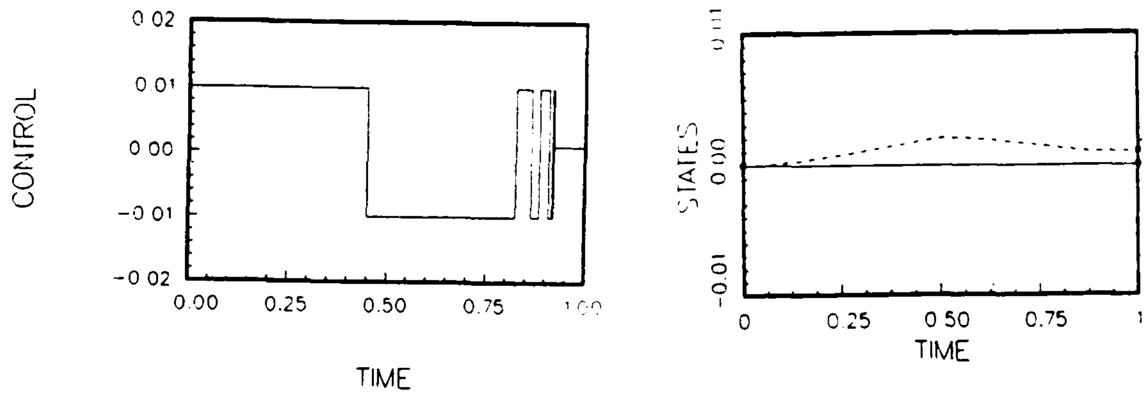
Clearly, for a boundary-value problem with a zero initial condition and small displacement to the end condition, an exact solution is nearly guaranteed when assuming a number of control legs equal to the order of the system (with the proviso that the end condition is physically attainable). This fact, coupled with the results already detailed in this section, points to a one-to-one correspondence between the control legs of the solutions and the system eigenvalues. It is as if each leg accounts for a single eigenvalue. For this reason, one can surmise that even a complicated 9th-order system can be retargeted with performance that is nearly optimal by accounting only for the dominant eigenvalues. In this case, the pole pair P_1 is clearly the dominant eigenvalue for the system as the closest eigenvalue has a time constant that differs by a factor of 100. The solution for the 2nd-order model should work reasonably well for the full-order model. Table 17 shows that this is in fact the case.

TABLE 6. Reduced-order solutions for the 9th-order problem.

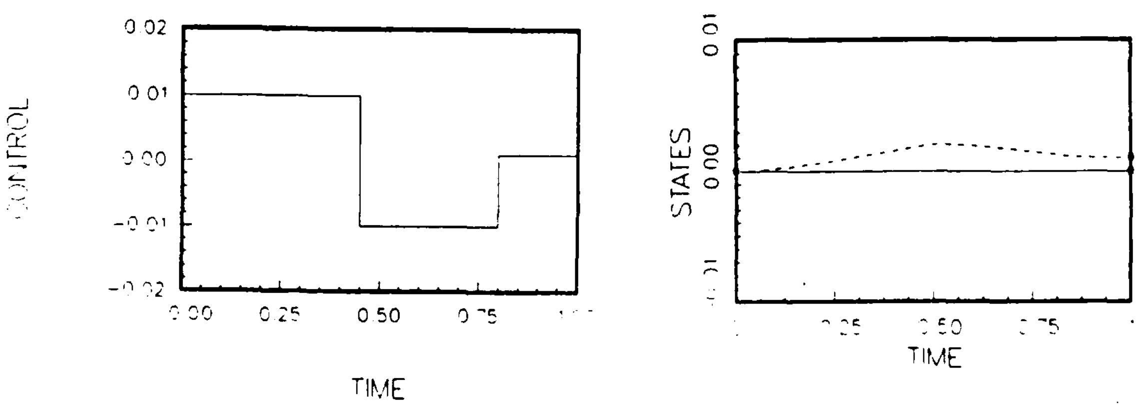
Model	$\lvert u \rvert$	x_1 at end of control (target = 0.00100)	Time of optimal control	Time required to settle to within 50 μrad of 0.001
2nd order	0.01	0.0012723	0.799	0.871
	0.005	0.0011486	1.119	1.171
	0.002	0.0010743	1.752	1.778
3rd order	0.01	0.0010427	0.861	0.859
	0.005	0.0010187	1.183	1.168
	0.002	0.001003	1.829	1.778
5th order[a]	0.01	0.0010096	0.888	0.862
	0.005	0.0010048	1.209	1.168
	0.002	0.0010017	1.850	1.778
7th order[a]	0.01	0.001000	0.920	0.861
	0.005	0.001000	1.256	1.168
	0.002	0.001000	1.879	1.778
9th order[a]	0.01	0.001000	0.922	0.861
	0.005	0.001000	1.240	1.168
	0.002	0.001000	1.878	1.777
Classical set-point	0.001			~16.6

[a] in higher-order problems, x_1 actually settles long before many of the other states.

It can be seen here that even the 2^{nd}-order solution, if it is followed by a steady-state input of u = 0.001, can achieve the desired end conditions at nearly the same time as the exact solution. Shown in Fig. 17 are the control profiles and time-responses of both the 2^{nd}order and the exact solution for $|u| = 0.01$.



(a) 2^{nd}-order solution.



(b) Exact solution.

Figure 17. Control profiles and time responses.

CONCLUSION

Time-optimal techniques offer a dramatic improvement in retargeting times for BSMs. A bang-bang time-optimal solution does this by accounting for each eigenvalue of a system and suppressing its resonances. A large-order system may be solved by starting from the most dominant eigenvalues and building a smaller order system. As more and more eigenvalues are added in, time-optimal solutions of the smaller systems are used to guess at the larger-order systems until an exact answer for the original system is found. However, the time-optimal solution can be approximated successfully with a lower-order system, if the lower-order system accounts for all of the critical eigenvalues.

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